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Unraveling *k*-Page Graphs

RAVI KANNAN*

Department of Computer Science, Carnegie–Mellon University, Pittsburgh, Pennsylvania 15213

I show in this note that for any integer k and any k page graph G, there is an easily constructed 3-page graph G' (called the unraveling of G) such that the minimum separator sizes of G and G' are within a factor of k of each other. Further the maximum degree of a vertex of G' is at most 2 plus the maximum degree of G. © 1985 Academic Press, Inc.

Outerplanar graphs are graphs that can be embedded on the plane such that all the vertices lie on the outer face of the embedding. Equivalently, the graph can be embedded on the plane so that all the vertices lie on one straight line and all the edges lie in one of the half planes defined by the line. Equivalently, a graph $G(V, E)$ on n vrtices is outerplanar if the vertices can be numbered 1, 2,..., n such that no two edges cross. (Two edges (i, j), and (k, l) cross if $i<k<j<l$ or $k<i<l<j$.) A k-page graph $G(V, E)$ (on n vertices) is a graph whose vertices can be ordered 1, 2,..., n and the edge set E partitioned into k sets $E_1, E_2, ..., E_k$ such that for each i no two edges of $G(V, E_i)$ cross each other. We say that such a graph has a k-page embedding. It is not difficult to see that 2-page graphs are planar. Conversely it has been shown recently by Buss and Shor (1984) that every planar graph can be embedded in eight pages. This number has been improved to seven by Heath (1984). Computation graphs of Turing machines are k-page graphs, where k depends only on the number of tapes of the Turing machine. This has been one reason for substantial interest in such graphs (e.g., see Pippenger, 1982; Paul, Pippenger, Szemeredi, and Trotter, 1983). These graphs also arise in connection with embeddings of VLSI circuits (Chung, Leighton, and Rosenberg, 1984) and fault tolerant arrays of processors. (Rosenberg, 1983) Intuitively, a k-page graph can be drawn on a "book" with k "pages" with all the vertices placed on the binding and no two edges on a page crossing. For this reason Buss and Shor (1984) call the minimum k for which a graph is k-page embeddable, the page number

* Department of Computer Science, Carnegie-Mellon University. Supported by NSF grant number MCS-8304770.

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of the graph. See also Baker (1983) and Syslo (1979) for a discussion of outerplanar graphs.

A *separator* of a graph $G(V, E)$ is a set of vertices whose removal leaves the graph in two or more connected components each with at most half[1] the number of vrtices of G. We denote by $b(G)$ the minimum cardinality of a separator of a graph G. A family of graphs $\mathscr{F}$ is said to be *separable* if there is a function $f(n) \in o(n)$ such that for any graph $G(V, E)$ in $\mathscr{F}$, there is a separator of cardinality at most $f(|V|)$. Outerplanar graphs are easily seen to be separable—indeed there is a set of two vertices that form a separator. Lipton and Tarjan (1980) showed that planar graphs have $O(\sqrt{n})$ separators and this of course applies to two-page graphs. For both applications to Turing machine problems and VLSI, it will be nice to prove or disprove the statement

For every fixed k, k-page graphs are separable.

The purpose of this note is to show that the above statement is true if and only if 3-page graphs are separable. Indeed we show that corresponding to a k-page graph $G(V, E)$ on n vertices there is a 3-page graph G on kn vertices such that

$$k \cdot b(G) \geqslant b(G') \geqslant b(G). \tag{1}$$

Suppose $G(V, E)$ is a k-page graph and $E_1, E_2, ..., E_k$ is a partition of the edges such that for some ordering of the vertex set V, the edges of $G(V, E_i)$ do not cross. Without loss of generality, let this ordering be $1, 2, ..., n$. Then G' has vertex set $V' = V \times \{1, 2, ..., k\}$ and the edge set E' of G' is defined as

$$E' = \bigcup_{i=1}^{k} \{((v, i), (u, i)): (v, u) \in E_i\}$$
$$\bigcup \{((v, i), (v, i+1)): v \in V \text{ and } 1 \leqslant i \leqslant k-1\}.$$

We show below that G' so defined is 3-page embeddable and satisfies inequalities (1). It may be viewed pictorially as an unraveling of the graph G and hence the title of the note. (This unravelling may be visualised in two steps: first the spine of the book is "thickened" by duplicating each vertex on each page and running a thread trough each copy of each vertex. Then the pages of the book are unhinged much as a carpenter's rule.)

LEMMA 1. *For every k-page graph $G(V, E)$, the corresponding G' defined above has page number at most* 3.

Proof. Let $G(V, E)$ be a k-page graph with n vertices $1, 2, ..., n$ and with

[1] We could equaly well use any fraction less than 1 for the purposes of this paper.

the partition $E_1, E_2, \ldots, E_k$ of its edges so that no two edges of $G(V, E_i)$ cross. We will order the kn vertices of G' by assigning a number $\sigma(v, i)$ between 1 and kn to each vertex (v, i) as follows:

$$\sigma(v, 2i+1) = 2in + v \qquad \text{for} \quad i = 0, 1, \ldots, \lfloor (k-1)/2 \rfloor$$

$$\sigma(v, 2i) = (2i-1)\,n + (n - v + 1) \qquad \text{for} \quad i = 1, 2, \ldots, \lfloor k/2 \rfloor$$

We partition the edge set E' of G' into three parts P, Q, R as follows:

$P = \{((v, i), (v, i+1)) : i \text{ is even}\}$
$Q = \{((v, i), (v, i+1)) : i \text{ is odd}\}$
$R = \{((v, i), (u, i)) : (v, u) \in E(G)\}$.

We wish to assert that $G'(V', S)$ has no two edges crossing for $S = P$, Q, R with the order on the vertices defined by σ, thus proving the lemma. $G'(V', R)$ is outerplanar by definition because σ orders the vertices $(1, i)$, $(2, i), \ldots, (n, i)$ in this order for odd i and exactly reverses the order for even i. Thus there are no crossing edges of R in the order defined by σ. For $G'(V', P)$ and $G'(V', Q)$, the reader can easily convince himself/herself that the assertion is valid by drawing a picture.

LEMMA 2. *Let $G(V, E)$ be a k-page graph and $G'(V', E')$ be as defined above. Then inequality* (1) *is satisfied.*

Proof. Suppose a set S of vertices forms a separator for the graph $G(V, E)$. Then consider the set of vertices

$$S' = \{(v, i) : v \in S;\ 1 \leqslant i \leqslant k\}.$$

It is not difficult to see that S' is a separator for G'. Indeed, if S_1 is the set of vertices in a connected component of $G \backslash S$ with $|S_1| \leqslant (\frac{1}{2})|V|$, then $S_1' = S_1 \times \{1, 2, \ldots, k\}$ form the vertices of a connected component of $G' \backslash S'$ and obviously satisfy the cardinality conditions. Conversely, suppose S' is a separator for the graph G'. Let

$$T' = \{(v, i) : (v, j) \in S' \text{ for some } j\}.$$

Then clearly, T' is still a separator of the graph G' and we have $|T'| \leqslant k \cdot |S'|$. Further $T' = T \times \{1, 2, \ldots, k\}$ for a suitable T of V. We wish to claim that T is a separator of $G(V, E)$. First, if on removing T, the graph G still remains connected, then on removing, T', the graph G' will remain connected—a contradiction. Further T_1 is the set of vertices in some connected component of $G \backslash T$ iff $T_1 \times \{1, 2, \ldots, k\}$ is the set of vertices in some connected component of $G' \backslash T'$. Thus the condition $|T_1| \leqslant (\frac{1}{2})|V|$ is implied by the condition $|T_1 \times \{1, 2, \ldots, k\}| \leqslant (\frac{1}{2})|V'|$ and we have proved the lemma.

We summarize the result in the following theorem:

THEOREM 1. *Corresponding to each k-page graph G there is an easily constructed 3-page graph G′ on kn vertices such that the minimum cardinalities of separators of G and G′ are within a factor of k of each other.*

k-page graphs of bounded degree are of much interest—indeed the family of computation graphs of any Turing machine have this property (where the bound on the degree depends only on the number of tapes). The construction above adds at most two to the maximum degree of any vertex, so the theorem is still valid for such grahs. More precisely,

THEOREM 2. *For any two integers k, l, the family of k-page graphs of valence at most l is separable iff the family of 3-page graphs with valence at most l + 2 is separable.*

We note the well-known theorem (Book, Greibach, and Wegbreit, 1970): every k-tape nondeterministic Turing machine can be simulated by a 2-tape nondeterministic one with at most a constant factor loss of time. The computation graphs of nondeterministic 2-tape Turing machines they get is very similar to the graph G' defined earlier.

A natural question to ask is whether similar results can be proved for pebbling (Hopcroft, Paul, and Valiant, 1977; Pippenger, 1980), i.e., can we show that if $f(n)$ pebbles are sufficient to pebble constant valence 3-page graphs on n vertices, they are sufficient for constant valence k-page graphs? Unfortunately, unraveling k-page graph G into a 3-page graph G' as above does not preserve the number of pebbles needed. The graph G' constructed above can be pebbled with $O(\log n)$ pebbles if it has n vertices—this is the case because G' is a levelled graph in k levels and each level can be pebbled with $O(\log n)$ pebbles. Using induction on k, it is not difficult to show that $O(\log n)$ pebbles suffice for the entire graph. However, it is known that in general k-page graphs require $\Omega(n/(\log n)^2)$ pebbles. (Paul, Tarjan, and Celoni, 1977 combined with Valiant, 1976.)

A similar situation holds for segregators defined in Paul, Pippenger, Szemerédi, and Trotter (1983). The unraveled version G' of any k-page graph G can be shown to have a $n^{(2k+1)/(2k+2)}$ segregator. (Segregators are for directed graphs—I mean here to take the natural directed unraveling.) It would be surprising if all k-page graphs had such small segregators though no result currently known rules this out. Schnitger (1983) has proved a general result from which he can conclude that general graphs of constant degree cannot have such small segregators.

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Preventing Unraveling in Social Networks: The Anchored k-Core Problem. Authors. Authors and affiliations.Â We provide polynomial-time algorithms for general graphs with k = 2, and for bounded-treewidth graphs with arbitrary k. We prove strong inapproximability results for general graphs and k â‰¥ 3. Keywords. Social Network General Graph Tree Decomposition Full Version Pure Nash Equilibrium. These keywords were added by machine and not by the authors. This process is experimental and the keywords may be updated as the learning algorithm improves. The phrase 'the unravelling of a plot' is used to describe how a writer builds up a story. It is commonly used when describing how the structure of the writing or the vocabulary used reveal the plot of the writing (commonly a novel or a biography). The plot of some writing is the story behind it - an example of this is 'the Lord of the Rings', in which a fellowship of people go on a quest to destroy the 'One Ring'. That is the plot of the story. The plot can be unravelled using a series of subplots. In the Lord of the Rings, subplots include the heroes travelling Create comics and graphic novels that jump off the screen. DA Muro. Paint a picture. Experiment with DeviantArtâ€™s own digital drawing tools. unraveling-theworld. ðŸŒ¹. 4 Watchers2K Page Views2 Deviations. unraveling-theworld. About Home Gallery Favourites Posts Shop. Send Note. Watch. About Me Statistics Watchers4 Watching17 Group Member1 Badges88 Comments. unraveling-theworld. March 13, 1991. United States.Â unraveling-theworld is not a Group Admin yet. Groups they admin or create will appear here. Group Member.