Rendiconti di Matematica, Serie VII
Volume 25, Roma (2005), 1-16

Transformation groups and submanifold geometry

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Abstract: *In the talk I give a survey on polar actions and generalizations of isoparametric hypersurfaces in space forms to more general ambient spaces.*

1 – Introduction

In this talk we will give a brief survey on generalizations of isoparametric hypersurfaces to submanifolds with higher codimension in various types of ambient spaces. We will also discuss the question when such submanifolds are homogeneous and introduce the isometric actions which have them as orbits.

A hypersurface M^n of a Riemannian manifold V^{n+1} is called *isoparametric* if M^n is locally a regular level set of a function f with the property that both $\| \operatorname{grad} f \|^2$ and Δf are constant on the level sets of f. One can show that M^n is an isoparametric hypersurface of V^{n+1} if and only if M^n and its parallel hypersurfaces have constant mean curvature.

The term 'isoparametric hypersurface' is due to Levi-Civita ([37]) and refers to the fact that $\| \operatorname{grad} f \|^2$ and Δf were at the time called the *first* and the *second differential parameter of f* respectively.

If the ambient space V^{n+1} is a real space form, then M^n can be shown to be an isoparametric hypersurface if and only if it has constant principal curvatures; see [9]. This characterization does not hold in more general ambient spaces; see [60] where counterexamples are given in complex projective spaces.

Key Words and Phrases: *Polar actions – Isoparametric submanifolds – Equifocal submanifolds.*
A.M.S. Classification: 53C30, 53C35, 53C40

Beniamino Segre proved the following theorem in [48]: let M^n be an isoparametric hypersurface in $\mathbf{R}^{n+1}$. Then M^n is a piece of a plane, of a sphere, or of a round cylinder. In particular it follows that M^n is homogeneous if it is complete. Conversely, it is clear that homogeneous hypersurfaces of $\mathbf{R}^{n+1}$ are isoparametric.

The case $n=2$ of the theorem of Segre was first proved by Somigliana ([49]) and later reproved in [47] and [37].

Cartan classified isoparametric hypersurfaces in hyperbolic spaces in [9] which also turn out to be homogeneous. He then turned to isoparametric hypersurfaces in spheres, see [10], [11], and [12], and noticed that the problem is much more difficult there than in the other real space forms. In [11] he asked three basic questions on isoparametric hypersurfaces in spheres. One of this questions was whether isoparametric hypersurfaces in spheres are homogeneous. A negative answer to Cartan's question was only given much later by Ozeki and Takeuchi in [42] who found inhomogeneous isoparametric hypersurfaces in spheres. These examples were later generalized by Ferus, Karcher and Münzner in [24].

I will not try to go further into the rich and beautiful theory of isoparametric hypersurfaces in spheres and refer to [58] for further information. Still I would like to mention the two highlights of the theory after the work of Cartan. The first are the papers [39] and [40] of Münzner where it is shown that the number g of principal curvatures of such a hypersurface can only be 1, 2, 3, 4 or 6. All of these numbers are known to occur. The second is the paper [50] of Stolz in which the possible multiplicities of the principal curvatures are determined. The contributions of Münzner and Stolz are important steps on the way to a full classification of isoparametric hypersurfaces in spheres, which is still an open problem.

2 – Polar actions

In this section we will discuss polar actions. The geometry of their principal orbits will serve us as a motivation in the generalizations of isoparametric hypersurfaces that we will present in the later sections.

Let V be a complete Riemannian manifold and let G be a Lie group acting on V by isometries. One says that the action is *polar* if there is a complete immersed submanifold Σ in V which meets all orbits of G in such a way that all intersections between Σ and orbits are perpendicular. The submanifold Σ is called a *section* of the action. It is rather easy to see that a section is totally geodesic; see [44] and [45], p. 95. The action is called *hyperpolar* if the section is flat.

One should think of a section as a set of canonical forms for the polar action as will be clear in the examples.

Example 2.1.

(i) Any isometric action with a hypersurface as an orbit is polar since a geodesic which meets one orbit orthogonally meets all orbits orthogonally.

(ii) Let V be the linear space $\mathcal{S}_0(n)$ consisting of real symmetric $n \times n$-matrices with zero trace endowed with the scalar product

$$\langle X, Y \rangle = \operatorname{trace}(XY).$$

Let G be the group $\mathrm{SO}(n)$ acting on V by conjugation. We let Σ denote the diagonal matrices in V. Then we know from linear algebra that every matrix X in V can be conjugated into Σ by an element of G. It is now an easy calculation to show that the intersections of conjugacy classes of matrices in V with Σ are all perpendicular. The action is therefore hyperpolar.

(iii) Let V be a compact connected Lie group G with a bi-invariant Riemannian metric acting on itself by conjugation. Let Σ be a maximal torus in G. The theorem on maximal tori says that all conjugacy classes in G meet Σ. An easy calculation shows that the intersections between conjugacy classes in G and Σ are all perpendicular. It follows that the action is hyperpolar since Σ is flat.

(iv) We now show how the examples (ii) and (iii) fit into the theory of symmetric spaces.

A *symmetric space* is a Riemannian manifold V such that for every point p in V there is an isometry σ_p of V fixing p and reversing the directions of the geodesics through p. We refer to the book [33] for what we will need from the theory of symmetric spaces. It is easy to show that symmetric spaces are homogeneous with respect to the isometry group. We can therefore write $V = G/K$ where G is the identity component of the isometry group of V and K is its isotropy group at some fixed point p_0 in V. Such a pair of groups (G, K) is called a *symmetric pair*.

Let Σ be a maximal flat and totally geodesic submanifold passing through p_0 in the symmetric space V. Then the action of K on V is hyperpolar with Σ as a section; see [32]. This example generalizes the one in (ii) since a compact connected Lie group K with a bi-invariant Riemannian metric is a symmetric space with a maximal torus as a maximal flat and totally geodesic submanifold. We can identify K with $K \times K/\Delta(K)$ where $\Delta(K)$ is the diagonal in $K \times K$ and it turns out that conjugation in K corresponds to the action of $\Delta(K)$ on $K \times K/\Delta(K)$.

One can generalize the action of K on the symmetric space $V = G/K$ as follows. Assume that (G, K_1) and (G, K_2) are symmetric pairs. Then one can show that the action of K_1 on $V = G/K_2$ is hyperpolar. This example was introduced by Hermann in [34] and we will refer to it as a *Hermann action*[1]. One gets concrete examples of this kind by considering

[1] Hermann proved in [34] that his examples are variationally complete and not that

Grassmann manifolds $G_k(\mathbf{C}^n) = \mathrm{SU}(n+1)/K_k$ where K_k is the stabilizer of $\mathbf{C}^k$ in $\mathbf{C}^n$. Then the actions of the groups $K_1, \dots, K_{n-1}$ on $G_k(\mathbf{C}^n)$ are all hyperpolar.

Now the action of K on V induces an action of K on the tangent space $T_{p_0}V$ which is called the *isotropy representation of the symmetric space* V. This isotropy representation is hyperpolar with $T_{p_0}\Sigma$ as a section. The example in (ii) is a special case and corresponds to the symmetric space $V = G/K$, where $G = \mathrm{SL}(n, \mathbf{R})$ and $K = \mathrm{SO}(n)$. One clearly has the following direct sum decomposition

$$\mathfrak{sl}(n, \mathbf{R}) = \mathfrak{so}(n) \oplus \mathcal{S}_0(n)$$

into skew and symmetric matrices, and this decomposition is invariant under $\mathrm{Ad}_G(K)$. Hence one can identify $T_{p_0}V$ with $\mathcal{S}_0(n)$ and the scalar product on $\mathcal{S}_0(n)$ in (ii) extends to G-invariant Riemannian metric on V. The action in (ii) now corresponds to the isotropy representation of $\mathrm{SL}(n, \mathbf{R})/\mathrm{SO}(n)$.

(v) We finally give an example of a polar action which is not hyperpolar. We let V be the complex projective space $P^n(\mathbf{C})$ endowed with the Fubini-Study metric which is invariant under the action of $\mathrm{SU}(n+1)$. Now let T^n be the maximal torus in $\mathrm{SU}(n+1)$ consisting of diagonal matrices. Then it is not difficult to see that the action of T^n on $P^n(\mathbf{C})$ is polar with $P^n(\mathbf{R})$ as a section. This action is of course not hyperpolar since any two sections of a polar actions are isometric and there can therefore not be a flat section.

The complex projective space $P^n(\mathbf{C})$ with the Fubini-Study metric is an example of a rank one symmetric space. Polar actions on compact rank one symmetric spaces were classified in [46]. It turns out that the sections are always real projective spaces if their dimension is at least two.

The following two theorems show that some of the examples above describe in fact the most general situation. We will need the concept of orbit equivalent actions in the statement of the theorems. Let K_1 act isometrically on V_1 and let K_2 act isometrically on V_2. Then the actions of K_1 and K_2 are said to be *orbit equivalent* if there is an isometry $f : V_1 \to V_2$ such that $f(K_1 p) = K_2 f(p)$ for all p in V_1, i.e., the orbits of K_1 and K_2 correspond under f.

Theorem 2.2 (Dadok [17]). *Let K be a compact group acting in a polar fashion on a Euclidean space V. Then the action of K is orbit equivalent to the isotropy representation of some symmetric space.*

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they are hyperpolar. The relationship between the two concepts will be explained at the end of this section.

The *cohomogeneity* of an action is the minimal codimension of its orbits. Eschenburg and Heintze gave in [21] a proof of Dadok's theorem under the assumption that the cohomogeneity is at least three. Their proof does not use the classification of compact Lie groups. Lists of polar representations that are not isotropy representations of symmetric spaces can be found in [4], [20], and [25].

THEOREM 2.3 (Kollross [36]). *Let $V = G/K$ be a compact irreducible symmetric space and let H be a subgroup of G which acts in a hyperpolar fashion on V with cohomogeneity at least two. Then the action of H on V is orbit equivalent to a Hermann action.*

Kollross also classifies in [36] all cohomogeneity one actions on compact irreducible symmetric spaces V. The classification of such actions on spheres was already carried out in [35].

We now discuss the principal orbits of polar and hyperpolar actions from the point of view of submanifold geometry. This will serve as a motivation for the generalizations of isoparametric hypersurfaces in the later sections.

Let G be a Lie group acting by isometries on a Riemannian manifold V. A *principal orbit* of the action of G on a manifold V is by definition an orbit Gp with the property that there is a neighborhood U of p such that there is a G-equivariant map from Gp to Gq for all q in U. If Gp is principal, then p is said to be *regular*. The set of regular points is open and dense in V. Now let ξ_0 be an element of $\nu_p(Gp)$ where $\nu(Gp)$ denotes the normal bundle of Gp. Then $\xi(gp) = dg_p(\xi_0)$ is a well defined normal vector field if Gp is principal. We call such a normal vector field *equivariant*.

For a proof of the following proposition, see [45], p. 95-96, or [5], p. 44.

PROPOSITION 2.4. *Assume that the action of G on V is polar. Then the equivariant normal vector fields along a principal orbit Gp are parallel. In particular, the normal bundle is flat and has trivial normal holonomy.*

The next property of the principal orbits of polar actions that we would like to present has to do with focal points. Let M be a submanifold of the Riemannian manifold V and assume that γ is a geodesic that starts in M, i.e. $\gamma(0)$ lies in M, and that $\gamma'(0)$ is perpendicular to M. Suppose $\gamma_s(t)$ is a smooth variation of $\gamma = \gamma_0$ such that $\gamma_s(0) \in M$ and $\gamma_s'(0)$ is perpendicular to M for all s. Now let J be the variational vector field

$$J(t) = \frac{\partial}{\partial s}\bigg|_{s=0} \gamma_s(t)$$

of γ_s. We call such a variational vector field an *M-Jacobi field along γ*. One can show that the M-Jacobi fields along γ form a vector space. A point $\gamma(t_0)$

is called a *focal point of M along* γ if there is a nonvanishing M-Jacobi field J with $J(t_0) = 0$. The dimension of the space of M-Jacobi fields vanishing in t_0 is called the *multiplicity* of the focal point $\gamma(t_0)$.

PROPOSITION 2.5. *Assume that the action of G on V is polar and let M be a principal orbit of G. Let ξ be a parallel normal field along M. Then the distances to the focal points and their multiplicities along the geodesic starting in direction $\xi(p)$ does not depend on p.*

If V is a Euclidean space then the focal points of M are determined by the principal curvatures. Let ξ be a normal vector field along M and X a tangent vector of M at p. We let $D_X\xi$ denote the directional derivative of ξ in direction X and denote the tangent component of $-D_X\xi$ by $A_\xi(X)$. It turns out that the map $A_\xi : T_pM \to T_pM$ that sends X to $A_\xi(X)$ is a selfadjoint linear endomorphism that depends only on the value of ξ at p. One calls A_ξ the *shape* (or *Weingarten*) *operator* of M in direction ξ_p. The eigenvalues of A_ξ are called the *principal curvatures of M in direction* ξ_p.

Now if ξ_p is a normal vector of M at p and λ is a nonvanishing principal curvature in direction ξ_p, then $p + (1/\lambda)\xi_p$ is a focal point of M along the line $\gamma(t) = p+t\xi_p$. Conversely if $p+(1/\lambda)\xi_p$ is a focal point of M along $\gamma(t) = p+t\xi_p$, then λ is a principal curvature in direction ξ_p.

We can therefore reformulate Proposition 2.5 as follows if the ambient space is Euclidean. Notice that a polar action on a Euclidean space is hyperpolar since the sections are affine subspaces.

PROPOSITION 2.6. *Let V be a Euclidean space on which a Lie group acts in a polar fashion. Let M be a principal orbit of G and let ξ be a parallel normal field along M. Then the principal curvatures in direction ξ_p do not depend on p.*

Before we end this section we would like to mention two classes of actions that are closely related to hyperpolar actions.

Variationally complete actions were introduced by BOTT in [6]; see also [7]. By definition an isometric action of a Lie group G on a Riemannian manifold V is called *variationally complete* if the following holds for all orbits M of G: let J be an M-Jacobi field along $\gamma(t)$ which vanishes at some point t_0. Then J is the variational vector field of a variation of the type $\phi_s(\gamma(t))$ where ϕ_s is a one-parameter subgroup of G. In other words, J is the restriction of a Killing field induced by the action of G to γ.

Conlon proved in [16] that a hyperpolar action on a complete Riemannian manifold is variationally complete. A partial converse was proved in [27]: a variationally complete action on a compact symmetric space is hyperpolar. It was previously proved by DI SCALA and OLMOS in [18], see also [25], that variationally complete representations are polar. Lytchak has conjectured that variationally complete actions on compact Riemannian manifolds with nonnegative sectional curvature are hyperpolar.

Variationally complete actions were introduced in [6] and [7] to study the Morse theory of geodesics on complete Riemannian manifolds and in particular on compact symmetric spaces. We next briefly review one of the main results of these papers.

Let M be a properly embedded submanifold of a Riemannian manifold V and p some point in V. We let $\mathcal{P} = \mathcal{P}(V, p \times M)$ denote the space of absolutely continuous paths $\gamma : [0, 1] \to V$ that start in p and end in M and for which the so-called *energy*

$$E(\gamma) = \int_0^1 \| \gamma'(t) \|^2 \, dt$$

is finite. Then $\mathcal{P}$ is in a natural way a Hilbert manifold and E is a smooth functional on $\mathcal{P}$, see [43], whose critical points are the geodesics starting in p and meeting M perpendicularly. If p is not a focal point of M, then the energy functional E is a Morse function in the sense that it has only nondegenerate critical points. We say that the submanifold M is *taut* if the energy functional is perfect, meaning that the number of critical points of index k of E in $\mathcal{P}$ is equal to the k-th Betti number of $\mathcal{P}$ with respect to $\mathbf{Z}_2$-coefficients, or equivalently, that the $\mathbf{Z}_2$-Morse inequalities of E on $\mathcal{P}$ are equalities; see [56]. An isometric action is called *taut* if all of its orbits are taut.

One of the main theorems of Bott and Samelson in [7] can now be phrased in our terminology by saying that variationally complete actions are taut.

A taut action does not have to be variationally complete. It is proved in [25], [26], and [28] that there are precisely three irreducible taut representations of compact groups which are not variationally complete. These three representations happen to be precisely the cohomogeneity three representations which are not variationally complete.

3 – Isoparametric submanifolds of Euclidean spaces

Isoparametric submanifolds in Euclidean spaces with higher codimension were first introduced by Harle in [29]. Carter and West independently introduced and studied such submanifolds with codimension three in [13] and [14]. Terng then dealt with the case of general codimension in [52].

According to [52] a complete and connected submanifold M^n in $\mathbf{R}^{n+k}$ is called *isoparametric* if its normal bundle is flat and if the principal curvatures in the direction of any (locally defined) parallel normal vector field are constant. It is proved in [52] that the normal holonomy of M^n is trivial. A locally defined parallel normal curvature vector can therefore be extended to a globally defined one.

It is proved in [52] that a noncompact isoparametric submanifold is the product embedding of a compact isoparametric one with a Euclidean space. We will therefore always assume compactness in the following. A compact isoparametric submanifold is contained in a round hypersphere; see [52]. We can always assume that M^n is not contained in any proper affine subspace. Such submanifolds are called *full*. An isoparametric submanifold is said to be *irreducible* if it cannot be nontrivially written as the product embedding of two isoparametric submanifolds.

Propositions 2.4 and 2.6 imply that principal orbits of polar representations are isoparametric. Conversely, Palais and Terng proved in [44] that a homogeneous isoparametric submanifold is such an orbit. One can show that an isoparametric hypersurface M^n in S^{n+1} is isoparametric in $\mathbf{R}^{n+2}$. The inhomogeneous examples of Ferus, Karcher and Münzner in [24] that we already mentioned in the introduction therefore give us a further class of examples. All known examples of irreducible isoparametric submanifolds in Euclidean spaces belong to one of these two classes of examples.

Terng developed a beautiful structure theory of isoparametric hypersurfaces in [52]. We would like to mention some of these results since they have been a paradigm in the generalizations.

Let M^n be an isoparametric submanifold in $\mathbf{R}^{n+k}$ and let ξ be a parallel normal field along M^n. The *end-point map in direction* ξ is the map $\eta_\xi : M^n \to \mathbf{R}^{n+k}$ one gets by setting $\eta_\xi(p) = p + \xi_p$. It turns out that the image of M^n under η_ξ that we denote by M_ξ is either a submanifold of dimension n or one of a lower dimension. We call M_ξ the *parallel submanifold of* M^n *in direction* ξ. If the dimension of M_ξ is equal to that of M^n, then M_ξ is also isoparametric and η_ξ is a diffeomorphism between M^n and M_ξ. If the dimension of M_ξ is smaller than that of M^n, then M_ξ consists of focal points of M^n and η_ξ is a fibration from M^n onto M_ξ. In this case we will call M_ξ a *focal submanifold.* One can show that the set F of focal points of M^n is precisely the union over the focal submanifolds of M^n.

It is easy to see with help of Proposition 2.4 that if M^n is homogeneous and hence a principal orbit of a polar representation, then the parallel submanifolds are nothing but the other orbits of the representation.

If M^n is isoparametric, then $\mathcal{F} = \{M_\xi \,|\, \xi \text{ parallel along } M^n\}$ is a family of disjoint submanifolds that covers the whole ambient space $\mathbf{R}^{n+1}$. It is not difficult to show that the isoparametric submanifolds in $\mathcal{F}$ foliate $\mathbf{R}^{n+1} \setminus F$, the complement of the focal set F of M^n. One can in fact show much more than this: $\mathcal{F}$ is a singular Riemannian foliation in the sense of Molino. This is a consequence of a much more general result of Töben in [59] that we will explain in the last section; see also [45], Corollary 8.5.6.

Terng associated in [52] a Coxeter group to an isoparametric submanifold M as follows. Let $\nu_p M$ be the normal space of M at p considered as an affine subspace of $\mathbf{R}^{n+k}$ and consider the set $F_p = F \cap \nu_p M$ of focal points contained in $\nu_p M$. Then F_p is a finite union over hyperplanes in $\nu_p M$ and the reflections in this hyperplanes generate a finite Coxeter group W that leaves the set F_p invariant. It then follows that the orbit of p under W is the intersection $M \cap \nu_p M$. The Coxeter group is implicit in Cartan's work for the codimension two case $M^n \subset S^{n+1} \subset \mathbf{R}^{n+1}$ since he proved that the focal points on the normal great circles to M^n in S^{n+1} are equidistant. In the codimension three case the Coxeter group was already found by Carter and West in [13].

The following theorem proved in [57] shows that isoparametric submanifolds come close to characterize principal orbits of polar representations.

Theorem 3.1. *Let M^n be an irreducible, full and compact isoparametric submanifold in M^{n+k} with $k \geq 3$. Then M^n is a principal orbit of a polar representation.*

Theorem 3.1 combined with Dadok's Theorem 2.2 gives a classification of irreducible isoparametric submanifolds with codimension at least three. The examples of Ferus, Karcher and Münzner are inhomogeneous with codimension $k = 2$. If the codimension is $k = 1$, then the round spheres are the only examples.

A new proof of Theorem 3.1 was given by Olmos in [41] using his theory of the normal holonomy of submanifolds; see also [5], Section 7.3. A further proof was given by Heintze and Liu in [30] as a byproduct of a proof of an analogous theorem in Hilbert spaces that will play a role in the next section. Eschenburg gave a proof of the theorem in [19] that uses Lie triple products.

4 – Equifocal submanifolds

In the paper [55], equifocal submanifolds of compact symmetric spaces were introduced and their basic theory developed as a generalization of isoparametric hypersurfaces in spheres and an analogue of the isoparametric submanifolds in Euclidean spaces. For symmetric spaces see reference [33] and the remarks in Example 2.1 (iv) above.

The definition of an equifocal submanifold is based on the properties of principal orbits of polar actions in Propositions 2.4 and 2.5.

Let M^n be a compact submanifold of a compact symmetric space V^{n+k}. We say that M^n is *equifocal* if the following three conditions are satisfied:

(i) The normal bundle of M^n is flat and has trivial holonomy.
(ii) If ξ is a parallel normal vector field and $\eta_\xi(p_0) = \exp(\xi(p_0))$ is a multiplicity k focal point of M^n for some p_0 in M^n, then $\eta_\xi(p) = \exp(\xi(p))$ is a multiplicity k focal point of M^n for all p in M^n. (In other words, the focal data of M^n are invariant under normal parallel translation.)
(iii) The image $\exp(\nu_p(M^n))$ of the normal space $\nu_p(M^n)$ of M^n at p is contained in some flat of V^{n+k} for all p in M^n.

Principal orbits of polar actions satisfy conditions (i) and (ii) in the definition above, and all three conditions are satisfied for principal orbits of hyperpolar actions.

The third condition is of course always satisfies if M^n is a hypersurface. It follows from [31] that a hypersurface M^n in a compact symmetric space V^{n+1} is equifocal if and only if it is isoparametric in the sense of the definition given at the beginning of this paper. One can of course define equifocal hypersurfaces in more general ambient spaces than symmetric spaces; see the next section. If the ambient space has nonpositive sectional curvature one should take into account that there might be focal points 'beyond infinity'; see [22]. It is not to be expected that such generalizations are equivalent to the concept of an isoparametric hypersurface if the ambient space is not symmetric.

In [55] we show that if the compact symmetric space V^{n+1} is irreducible, then an equifocal hypersurface M^n in V^{n+1} has the property that any geodesic meeting M^n is closed. If V^{n+1} is simply connected, then one can show that the number of focal points on such a normal closed geodesic is an even number that we will denote by $2g$. If V^{n+1} is a sphere, then g is the number of principal curvatures of M^n which can only be one of the numbers 1, 2, 3, 4, and 6 as was proved by Münzner; see the introduction. One can now ask which values g can assume in general irreducible symmetric spaces, and what the possible values of the multiplicities of the focal points are; see [51] and [23] where this question is studied for rank one and two symmetric spaces.

One can prove more generally that the image of a normal space $\nu_p(M)$ of an equifocal submanifold M^n in an irreducible compact symmetric space V^{n+k} is a torus T^k; see [55]. One can associate to M^n an affine Coxeter group as follows. Let F denote the set of focal points of M^n in T^k and let $\tilde{F}$ be the preimage of F in the universal cover $\mathbf{R}^k$ of T^k. Then $\tilde{F}$ is a union of hyperplanes which are precisely the mirrors of an affine Coxeter group W acting on $\mathbf{R}^k$ and leaving $\tilde{F}$ invariant.

The next theorem which is analogous to Theorem 3.1 gives a characterization of the principal orbits of hyperpolar actions as equifocal submanifolds.

THEOREM 4.1 (Christ [15]). *Let M^n be an equifocal submanifold in an irreducible compact symmetric space V^{n+k}. Then M^n is the principal orbit of a hyperpolar action if $k \geq 2$.*

The theorem does not hold for $k = 1$ since the inhomogeneous isoparametric hypersurfaces in spheres are equifocal.

Theorem 4.1 and the results from [55] that we have been explaining are proved with the help of a generalization due to Terng of the theory of isoparametric submanifolds in Euclidean spaces to Hilbert spaces; see [53]. We end this section with a short explanation of this method.

Let V^{n+k} be a compact symmetric space that we write as a coset space $V^{n+k} = G/K$ where (G, K) is a symmetric pair. Let $\mathfrak{g}$ denote the Lie algebra of G and set $\mathcal{H} = L^2([0,1], \mathfrak{g})$, the Hilbert space of L^2-paths in $\mathfrak{g}$. Then there is a Riemannian submersion $\phi : \mathcal{H} \to V^{n+k}$ such that a submanifold M^n is equifocal in V^{n+k} if and only if the preimage $\mathcal{M} = \phi^{-1}(M^n)$ is isoparametric in $\mathcal{H}$. The main point is that it is easier to work in $\mathcal{H}$ than in V^{n+1} since $\mathcal{H}$ is linear, although infinite dimensional.

To define the Riemannian submersion ϕ, we need to introduce certain path spaces in G. Let B be a subset of $G \times G$ and let $\mathcal{P}(G, B)$ denote the space of absolutely continuous paths $\gamma : [0,1] \to G$ such that $(\gamma(0), \gamma(1)) \in B$ and such that the integral $\| \gamma' \|^2$ is finite. Here we assume G to be endowed with a bi-invariant Riemannian metric such that the projection $\pi : G \to V^{n+k}$ is a Riemannian submersion. Then $\mathcal{P}_e = \mathcal{P}(G, e \times G)$ is the space of paths in G starting at the identity e without a restriction on the end point.

Now it turns out that the map that sends a path γ in $\mathcal{P}_e$ to $\gamma^{-1}\gamma'$ in $\mathcal{H}$ is a diffeomorphism. Let $E : \mathcal{H} \to \mathcal{P}_e$ denote the inverse of this diffeomorphism. Now we can define a map $\psi : \mathcal{H} \to G$ by setting $\psi(u)$ equal to the endpoint of the curve $E(u)$, i.e. $\psi(u) = E(u)(1)$. It is proved in [55] that ψ is a Riemannian submersion. Now we define $\phi : \mathcal{H} \to V^{n+k}$ as $\phi = \pi \circ \psi$.

If H is a subgroup of G then $\mathcal{P}(G, H \times K)$ is an infinite dimensional Hilbert Lie group which acts on $\mathcal{H}$ by setting

$$\gamma * u = \gamma u \gamma^{-1} - \gamma' \gamma^{-1}$$

for γ in $\mathcal{P}(G, H \times K)$ and u in $\mathcal{H}$; see [54] where it is proved that the action of H on V^{n+k} is hyperpolar if and only if the action of $\mathcal{P}(G, H \times K)$ is polar on $\mathcal{H}$. It is also proved in [54] that the principal orbits of $\mathcal{P}(G, H \times K)$ are isoparametric if its action on $\mathcal{H}$ is polar.

A very important result of Heintze and Liu in [30] is that an irreducible isoparametric submanifold in an infinite dimensional Hilbert space is the principal orbit of a polar action if its codimension is at least two. This result of Heintze and Liu is one of the main steps in the proof of Theorem 4.1. The method of proof also works in finite dimensions if the codimension is at least three and can be used to prove Theorem 3.1.

One can also use the Hilbert space $\mathcal{H}$ to prove that an action on a compact symmetric space is hyperpolar if it is variationally complete; see [27] and Section 2. One shows that the action of a subgroup H of G is variationally complete (resp. hyperpolar) on V^{n+k} if and only if the action of $\mathcal{P}(G, H \times K)$ on $\mathcal{H}$ is variationally complete (resp. hyperpolar). One has now reduced the problem to an affine action on the linear space $\mathcal{H}$ and can argue in a similar way as as in [18].

5 – Submanifolds in Riemannian manifolds

In this last section we would like to mention some recent generalizations to Riemannian manifolds.

The orbits of a connected Lie group acting by isometries on a Riemannian manifold give an example of a singular Riemannian foliation in the sense of Molino; see [38], p. 189. By definition, a partition $\mathcal{F}$ of a Riemannian manifold V into connected immersed submanifolds, called *leaves*, is said to be a *singular Riemannian foliation* if the following two conditions are satisfied:

(i) The tangent space T_pM for every M in $\mathcal{F}$ and every p in M is generated by $\{X_p|\ X \in \Xi_{\mathcal{F}}\}$ where $\Xi_{\mathcal{F}}$ denotes the module of smooth vector fields on V that are tangent to the submanifolds in $\mathcal{F}$.

(ii) A geodesic that meets one leaf M in $\mathcal{F}$ perpendicularly, meets the leaves perpendicularly for all parameter values.

The leaves in $\mathcal{F}$ of maximal dimension are called *regular* and those of lower dimension *singular*.

If only the first condition is satisfied then one calls $\mathcal{F}$ a singular foliation. A singular foliation is a foliation in the usual sense if the leaves are all regular. The second condition means that the leaves are equidistant.

If $\mathcal{F}$ consists of the orbits of an action, then condition (ii) is satisfied since the vector fields it induces are contained in $\Xi_{\mathcal{F}}$ and condition (ii) is satisfied if the action is isometric.

Alexandrino studies singular Riemannian foliations that admit a section in [2], where a section is defined as for polar actions. Previously such foliations were studied by Boualem in [8]. Let $\mathcal{F}$ be such a singular foliation in a Riemannian manifold V, let L be a singular leaf in $\mathcal{F}$, and let T be a tubular neighborhood of L that is a union over leaves in $\mathcal{F}$. It is then proved in [2] that the foliation consisting of the intersections of the leaves of such a foliation $\mathcal{F}$ with the connected component of $\exp(\nu_p(L)) \cap T$ containing p is diffeomorphic to an isoparametric foliation in an open neighborhood of 0 in $\mathbf{R}^k$ where k is the codimension of L in V. This generalizes the slice theorems for polar actions and isoparametric submanifolds; see [45]. A further result of [2] is that the regular leaves of singular Riemannian foliations with a section have parallel focal structure, see also [59] for a different proof. Submanifolds with parallel focal structure were studied by Ewert in [22]. They generalize equifocal submanifolds in a similar way as polar actions generalize hyperpolar actions, see [59] for a precise definition. Töben gives in [59] a necessary and sufficient condition for a submanifold with parallel focal structure and finite normal holonomy to give rise to a singular Riemannian foliation with the leaves being parallel submanifolds. In [59] an action on the sections of a singular Riemannian foliations by a group called transversal holonomy group is introduced. This action generalizes the Weyl group action of polar actions.

In [1] Alexandrino studies transnormal maps. These are by definition maps from a Riemannian manifold into a Euclidean space with the property that its restrictions to sufficiently small neighborhoods of regular level sets are Riemannian submersions such that the normal spaces of the fibers form an integrable distribution on the neighborhood. The main result of [2] is that the level sets of an analytic transnormal map on a real analytic Riemannian manifold give rise to a singular Riemannian foliation with sections.

Further results along these lines can be found in [3].

REFERENCES

[1] M. Alexandrino: *Integrable Riemannian submersion with singularities*, to appear in Geometriae Dedicata.

[2] M. Alexandrino: *Singular Riemannian foliations with sections*, E-print math. DG/0311454, to appear in Illinois J. Math.

[3] M. Alexandrino – D. Töben: *Singular Riemannian foliations on simply connected spaces*, preprint 2004.

[4] I. Bergmann: *Reducible polar representations*, Manuscripta Math., **104** (2001), 309-324.

[5] J. Berndt – S. Console – C. Olmos: *Submanifolds and holonomy*, Research Notes in Mathematics, Vol. 434, Chapman & Hall/CRC, Boca Raton, 2003.

[6] R. Bott: *An application of the Morse theory to the topology of Lie-groups*, Bull. Soc. Math. France, **84** (1956), 251-281.

[7] R. Bott – H. Samelson: *Applications of the theory of Morse to symmetric spaces*, Amer. J. Math., **80** (1958), 964-1029, correction in Amer. J. Math., **83** (1961), 207-208.

[8] H. Boualem: *Feuilletages riemanniens singuliers transversalement intégrables*, Compositio Math., **95** (1995), 101-125.

[9] É. Cartan: *Familles de surfaces isoparamétriques dans les espaces à courbure constante*, Ann. di Mat., **17** (1938), 177-191.

[10] É. Cartan: *Sur des familles remarquables d'hypersurfaces isoparamétriques dans les espaces sphériques*, Math. Z., **45** (1939), 335-367.

[11] É. Cartan: *Sur quelques familles remarquables d'hypersurfaces*, C. R. Congrès Math. Liège, (1939), 30-41.

[12] É. Cartan: *Sur des familles d'hypersurfaces isoparamétriques des espaces sphériques à 5 et à 9 dimensions*, Revista Univ. Tucumán, **1** (1940), 5-22.

[13] S. Carter – A. West: *Generalised Cartan polynomials*, J. London Math. Soc., **32** (1985), 305-316.

[14] S. Carter – A. West: *Isoparametric systems and transnormality*, Proc. London Math. Soc., **51** (1985), 520-542.

[15] U. Christ: *Homogeneity of equifocal submanifolds*, J. Differential Geom., **62** (2002), 1-15.

[16] L. Conlon: *Variational completeness and K-transversal domains*, J. Differential Geom., **5** (1971), 135-147.

[17] J. Dadok: *Polar coordinates induced by actions of compact Lie groups*, Trans. Amer. Math. Soc., **288** (1985), 125-137.

[18] A. J. Di Scala – C. Olmos: *Variationally complete representations are polar*, Proc. Amer. Math. Soc., **129** (2001), 3445-3446.

[19] J.-H. Eschenburg: *Isoparametric submanifolds and symmetric spaces*, Geometry and topology: Aarhus (1998), Contemp. Math., Vol. 258, Amer. Math. Soc., Providence, RI, 2000, pp. 145-155.

[20] J.-H. Eschenburg – E. Heintze: *On the classification of polar representations*, Math. Z., **232** (1999), 391-398.

[21] J.-H. Eschenburg – E. Heintze: *Polar representations and symmetric spaces*, J. Reine. Angew. Math., **507** (1999), 93-106.

[22] H. Ewert: *Equifocal submanifolds in Riemannian symmetric spaces*, Ph.D. thesis, University of Cologne, 1998.

[23] F. Fang: *Equifocal hypersurfaces in symmetric spaces*, Chinese Ann. Math. Ser. B, **21** (2000), 473-478.

[24] D. Ferus – H. Karcher – H. F. Münzner: *Cliffordalgebren und neue isoparametrische Hyperflächen*, Math. Z., **177** (1981), 479-502.

[25] C. Gorodski – G. Thorbergsson: *Representations of compact Lie groups and the osculating spaces of their orbits*, preprint, University of Cologne, 2000 (also E-print math. DG/0203196).

[26] C. Gorodski – G. Thorbergsson: *Cycles of Bott-Samelson type for taut representations*, Ann. Global Anal. Geom., **21** (2002), 287-302.

[27] C. Gorodski – G. Thorbergsson: *Variationally complete actions on compact symmetric spaces*, J. Differential Geom., **62** (2002), 39-48.

[28] C. Gorodski – G. Thorbergsson: *The classification of taut irreducible representations*, J. Reine Angew. Math., **555** (2003), 187-235.

[29] C. E. Harle: *Isoparametric families of submanifolds*, Bol. Soc. Brasil. Mat., **13** (1982), 35-48.

[30] E. Heintze – X. Liu: *Homogeneity of infinite-dimensional isoparametric submanifolds*, Ann. of Math., **149** (1999), 149-181.

[31] E. Heintze – X. Liu – C. Olmos: *Isoparametric submanifolds and a Chevalley-type restriction theorem*, E-print math. DG/0004028.

[32] E. Heintze – R. S. Palais – C.-L. Terng – G. Thorbergsson: *Hyperpolar actions on symmetric spaces*, Geometry, Topology, and Physics for Raoul Bott, S. T. Yau (eds.), Conf. Proc. Lecture Notes Geom. Topology, IV, International Press, Cambridge, MA, 1995, pp. 214-245.

[33] S. Helgason: *Differential geometry, Lie groups, and symmetric spaces*, Pure and Applied Mathematics, Vol. 80, Academic Press, New York-London, 1978.

[34] R. Hermann: *Variational completeness for compact symmetric spaces*, Proc. Amer. Math. Soc., **11** (1960), 544-546.

[35] W.-Y. Hsiang – H. B. Lawson Jr.: *Minimal submanifolds of low cohomogeneity*, J. Differential Geometry, **5** (1971), 1-38.

[36] A. Kollross: *A classification of hyperpolar and cohomogeneity one actions*, Trans. Amer. Math. Soc., **354** (2002), 571-612.

[37] T. Levi-Civita: *Famiglie di superficie isoparametriche nell'ordinario spazio euclideo*, Rend. Acc. Naz. Lincei, **XXVI** (1937), 657-664.

[38] P. Molino: *Riemannian foliations*, Progress in Mathematics, Vol. 73, Birkhäuser, Boston-Basel, 1988.

[39] H. F. Münzner: *Isoparametrische Hyperflächen in Sphären,* I, Math. Ann., **251** (1980), 57-71.

[40] H. F. Münzner: *Isoparametrische Hyperflächen in Sphären,* II, Math. Ann., **256** (1981), 215-232.

[41] C. Olmos: *Isoparametric submanifolds and their homogeneous structures*, J. Differential Geom., **38** (1993), 225-234.

[42] H. Ozeki – M. Takeuchi: *On some types of isoparametric hypersurfaces in spheres, i*, Tôhoko Math. J., **27** (1975), 515-559.

[43] R. Palais: *Morse theory on Hilbert manifolds*, Topology, **2** (1963), 299-340.

[44] R. Palais – C.-L. Terng: *A general theory of canonical forms*, Trans. Amer. Math. Soc., **300** (1987), 771-789.

[45] R. Palais – C.-L. Terng: *Critical point theory and submanifold geometry*, Lecture Notes in Mathematics, Vol. 1353, Springer-Verlag, Berlin, Heidelberg, New York, 1988.

[46] P. Podestà – G. Thorbergsson: *Polar actions on rank-one symmetric spaces*, J. Differential Geom., **53** (1999), 131-175.

[47] B. Segre: *Una proprietà caratteristica di tre sistemi* ∞^1 *di superficie*, Atti Acc. Sc. Torino, **LIX** (1924), 666-671.

[48] B. Segre: *Famiglie di ipersuperficie isoparametriche negli spazi euclidei ad un qualunque numero di dimensioni*, Rend. Acc. Naz. Lincei, **XXVII** (1938), 203-207.

[49] C. Somigliana: *Sulle relazione fra il principio di Huygens e l'ottica geometrica*, Atti Acc. Sc. Torino, **LIV** (1918-1919), 974-979.

[50] S. Stolz: *Multiplicities of Dupin hypersurfaces*, Invent. Math., **138** (1999), 253-279.

[51] Z. Tang: *Multiplicities of equifocal hypersurfaces in symmetric spaces*, Asian J. Math., **2** (1998), 181-213.

[52] C.-L. Terng: *Isoparametric submanifolds and their Coxeter groups*, J. Differential Geom., **21** (1985), 79-107.

[53] C.-L. Terng: *Proper Fredholm submanifolds of Hilbert space*, J. Differential Geom., **29** (1989), 9-47.

[54] C.-L. Terng: *Polar actions on Hilbert space*, J. Geom. Anal., **5** (1995), 129-150.

[55] C.-L. Terng – G. Thorbergsson: *Submanifold geometry in symmetric spaces*, J. Differential Geom., **42** (1995), 665-718.

[56] C.-L. Terng – G. Thorbergsson: *Taut immersions into complete Riemannian manifolds*, Tight and taut submanifolds (Berkeley, CA, 1994), T. E. Cecil, S. S. Chern (eds.), Math. Sci. Res. Inst. Publ., Cambridge Univ. Press, Cambridge, 1997, pp. 181-228.

[57] G. Thorbergsson: *Isoparametric foliations and their buildings*, Ann. of Math., **133** (1991), 429-446.

[58] G. Thorbergsson: *A survey on isoparametric hypersurfaces and their generalizations*, Handbook of differential geometry, Vol. I, F. Dillen, L. Verstraelen (eds.), North-Holland, Amsterdam, 2000, pp. 963-995.

[59] D. Töben: *Parallel focal structure and singular Riemannian foliations*, E-print math. DG/0403050, to appear in Trans. Amer. Math. Soc.

[60] Q. M. Wang: *Isoparametric hypersurfaces in complex projective spaces*, Proceedings of the 1980 Beijing Symposium on Differential Geometry and Differential Equations, Vol. 1, 2, 3 (Beijing, 1980), Science Press, Beijing, 1982, pp. 1509-1523.

Lavoro pervenuto alla redazione il 14 ottobre 2004
ed accettato per la pubblicazione il 26 novembre 2004.
Bozze licenziate il 14 febbraio 2005

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This is considered to be an active coordinate transformation. Now, in my physics courses I've always learnt that active and passive transformations are exactly the same, and it is just a matter of convention which one we choose. However, the above seems to suggest that it is always possible to change coordinate via an active transformation, whereas it is only possible to make a passive coordinate transformation if at least two open sets overlap each other. Is this true? Or is is it also possible to somehow make an passive coordinate transformation if the open sets do not overlap. I guess,